Extremal and Probabilistic Graph Theory
Lecture 10
March 31st, Thursday

**Theorem 10.1** (Simonovits' Stability Theorem). *$\forall \varepsilon > 0$, and $\forall$ family $\mathcal{F}$ with $\chi(\mathcal{F}) = r+1$, $\exists \delta$ and $n_0$ s.t. if $G$ is $\mathcal{F}-free$ with $n \geq n_0$ vertices and $e(G) \geq (1-\frac{1}{r})\binom{n}{2} - \delta n^2$, then $d(G, T_r(n)) \leq \varepsilon n^2$*

We have proved just for $\mathcal{F} = \{K_{r+1}\}$; and the general case will be proved after *Regularity Lemma*.

**Definition 10.2.** A graph $F$ is $r$-edge-critical if there exists an edge $e$ such that $\chi(F-e) < \chi(F) = r$.

**Definition 10.3.** A $r$-partite graph $G$ with $V(G) = V_1 \cup V_2 \cup ... \cup V_r$ is $\varepsilon$-*almost complete* if $\forall i < j,\ x \in V_i,\ |N(x) \cap V_j| \geq |V_j| - \varepsilon n$, where $|V_i| >> \varepsilon n$.

**Lemma 10.4.** *Let $\varepsilon \in (0,1)$ small enough. Let $F$ be $(r+1)$-edge-critical with $b = |V(F)|$. Let $G$ be an $F-free$ graph containing an $\varepsilon$-almost complete $r$-partite spanning subgraph $G'$ with $V(G') = Z_1 \cup ... \cup Z_r$ where $|Z_i| >> \varepsilon n$. Then*

*(i). Each $Z_i$ is independent.*

*(ii). If we add an vertex $w$ to $G$ and get an $F$-free graph, then $\exists i$ s.t. $w$ has at most $\varepsilon brn$ neighbors in $Z_i$*

*Proof.* Let $b = |V(F)|$. Note that there exists an edge $xy \in E(F)$ such that $F - xy$ is $r$-partite.

For (i), suppose that some $Z_i$ (say $Z_1$) contains an edge $uv$. We then can find $B_i \subset Z_i$ with $|B_i| = b$ such that $\{uv\} \in B_1$ and $B_1, B_2, \cdots, B_r$ form a complete $k$-partite subgraph $H$ (this is left as an exercise). Then it's clear that $F \subset H \subset G$.

We consider (ii). Suppose for a contradiction that $w$ has at least $\varepsilon brn$ neighbors in each $Z_i$.

Claim: There are $B_i \subseteq Z_i$ with $|B_i| = b$ s.t. $B_1 \cup ... \cup B_r \cup \{w\}$ forms a complete $(r+1)$-partite $K_{b,b,...,b,1}$ in $G$.(Clearly $F \subseteq K_{b,b,...,b,1}$ so this is a contradiction)

First, choose $B_1 \subseteq Z_1 \cap N(w)$ with $|B_1| = b$ suppose we have $B_1 \subseteq Z_1, ..., B_i \subseteq Z_i$ s.t. $B_1 \cup ... \cup B_i \cup \{w\}$ forms $(i+1)$-partite $K_{b,b,...,b,1}$ $i \leq r-1$. Each $u \in B_1 \cup ... \cup B_i$ misses at most $\varepsilon n$ vertices in $Z_{i+1}$, thus there is a set $S$ of $Z_{i+1}$ with $|S| \geq |Z_{i+1}| - ib\varepsilon n$ such that each vertices in $S$ is adjacent to each vertex of $B_1 \cup ... \cup B_i$. But $w$ has$\geq rb\varepsilon n$ neighbors in $Z_{i+1}$. Therefore we can find $B_{i+1} \subseteq S \cap N(w)$ with $|B_{i+1}| = b$. This proves the claim. Then (ii) is complete. ∎

**An application of Stability approach**

**Theorem 10.5.** *Let $F$ be an $(r+1)$-edge-critical where $r \geq 2$. Then for sufficiently large $n$ (say $n \geq n_0(F)$), $ex(n, F) = e(T_r(n))$ and the unique extremal graph is $T_r(n)$*

*Proof.* Let $G$ be an $F$-free graph with $e(G) \geq e(T_r(n))$. Our goal is to show $G = T_r(n)$.

Claim: It sufficient to consider $G$ with an additional condition that $\delta(G) \geq \delta(T_r(n))$

*Proof of Claim.* If $G_n = G$ has a vertex $v_n$ of degree less than $\delta(T_r(n))$ then $G_{n-1} = G_n - \{v_n\}$ is s.t.

$$e(G_{n-1}) = e(G_n) - d(v_n) \geq e(T_r(n)) - \delta(T_r(n) + 1 = e(T_r(n-1)) + 1$$

Suppose we have defined $G_m$ with $e(G_m) \geq e(T_r(m)) + (n - m)$. If $G_m$ has a vertex $v_m$ with $d_{G_m}(v_m) < \delta(T_r(n))$, then $G_{m-1} = G_m - \{v_m\}$ and similarly we can show

$$e(G_{m-1}) \geq e(Tr(m-1)) + n - m + 1$$

This process must terminate at some step, say $G_t$(having $t$ vertices). Then

$$\binom{t}{2} \geq e(G_t) \geq e(T_r(t)) + n - t \geq n - t \Rightarrow t \geq \sqrt{n} \textit{ large enough}$$

Note that $\delta(G_t) \geq \delta(T_r(t))$. Now assume that under the additional condition $\delta(G_t) \geq \delta(T_r(t))$ one can prove $G_t = T_r(t)$

$$\Rightarrow e(T_r(t)) = e(G_t) \geq e(T_r(t)) + n - t \ \Rightarrow t = n$$

$$\Rightarrow G = G_n = G_t = T_r(n)$$

This proves the claim.

Take $\varepsilon$ to be small enough ($\varepsilon := \varepsilon(F)$)

By Stability Theorem, as $e(G) \geq e(T_r(n))$, then $d(G, T_r(n)) \leq \varepsilon n^2$. So there exists an $r$-partition $V_1 \cup ... \cup V_r$ of $G$ s.t.

$$\sum_{i=1}^{r} e(V_i) + \text{“}\textit{missing edges}\text{”} \geq \varepsilon n^2 \qquad (*)$$

where $|V_i| = \lceil \frac{n}{r} \rceil$ *or* $\lfloor \frac{n}{r} \rfloor$. Here, a *missing edge* is a pair $(x, y)$ with $x \in V_i, y \in V_j$ s.t. $xy \notin E(G)$. We say a vertex $u$ (say $u \in V_i$) is “bad” if $\exists V_j (j \neq i)$ $|N(u) \cap V_j| < |V_j| - \sqrt{\varepsilon} n$ (i.e. $u$ has$\geq \sqrt{\varepsilon} n$ missing edges).

Let $B = \{\textit{all bad vertices}\}$. Then

$$|B| \leq \frac{2\varepsilon n^2}{\sqrt{\varepsilon} n} = 2\sqrt{\varepsilon} n$$

Let $U_i = V_i \setminus B$ with

$$|U_i| \geq |V_i| - |B| \geq \frac{n}{r} - 2\sqrt{\varepsilon} n$$

And each $x \in U_i$ satisfies that

$$|N(x) \cap U_j| \geq |N(x) \cap V_j| - |B| \geq |V_j| - 3\sqrt{\varepsilon} n \geq |U_j| - 3\sqrt{(\varepsilon)} n$$

So $(U_1, ..., U_r)$ is a $3\sqrt{\varepsilon}$-almost complete $r$-partition. Let $\varepsilon' = 5\sqrt{\varepsilon}$.

By Lemma(i), each $U_i$ is independent. Consider each $x \in B$. Since $G[(\cup U_i) \cup \{x\}]$ is $F$-free, by Lemma(ii), $\exists$ some $U_i$ with $|N(x) \cap U_i| \leq \varepsilon' brn \leq 5\sqrt{\varepsilon} brn$

By claim, $d_G(x) \geq \delta(T_r(n))$. So

$$|N(x) \cap (\cup_j U_j)| \geq d_G(x) - |B| \geq n - \lceil \frac{n}{r} \rceil - 2\sqrt{\varepsilon} n$$

$$\Rightarrow |N(x) \cap (\cup_j U_j \setminus U_i)| \geq n - \frac{n}{r} - 7\sqrt{\varepsilon} brn \geq |\cup_j U_j \setminus U_i| - 7\sqrt{\varepsilon} brn$$

In particular, for $\forall j \neq i$, $|N(x) \cap U_j| \geq |U_j| - 7\sqrt{\varepsilon}brn$. We then add this $x$ into $U_i$ to get a new $r$-partition $(U_1', ..., U_r')$ which is $7\sqrt{\varepsilon}br$-almost complete.

By Lemma(i), $U_i'$ is independent, i.e. $N(x) \cap U_i = \emptyset$.

Then repeating the above process $\Rightarrow$ for $j \neq i, |N(x) \cap U_j| \geq |U_j| - 2\sqrt{\varepsilon}n$. So the new $(U_1', ..., U_i')$ is $\varepsilon'$-almost complete where $\varepsilon' = 5\sqrt{\varepsilon}$.

We can keep adding vertices in $B$ into the $r$-partition $(U_1, ..., U_r)$ using the operator $(*)$ which is always $5\sqrt{\varepsilon}$-almost complete until $B =$. Then in the end. $V(G) = U_1 \cup ... \cup U_r$ and by lemma(i) each $U_i$ is independent. So G is exactly $r$-partite. Since $e(G) \geq e(T_r(n))$ we see that $G = T_r(n)$(Because $T_r(n)$ is the unique graph achieving the max number edges among all $r$-partite graphs). ∎

Next, we introduce the concept of *Decomposition family.*

**Definition.** Given a graph $F$ with $\chi(F) = r$, its decomposition family $\mathcal{M} = \mathcal{M}_F$ is the set of bipartite graphs obtained from any proper r-coloring of $F$ by deleting any set of $r-2$ color classes of this coloring.

If $F$ =edge-critical, then a member of $\mathcal{M}_F$ consists of one edge.

**Exercise.** For any $F$ with $\chi(F) = r+1$, $ex(n, F) \geq e(T_r(n)) + z(\frac{n}{2r}, \frac{n}{2r}, \mathcal{M}_F)$.

Erdos-Stone-Simonovits tells us that $ex(n, F) = e(T_r(n)) + o(n^2)$, for $F$ with $\chi(F) = r+1$. The remainder $o(n^2)$ depends primarily on $\mathcal{M}_F$. This roughly says that the general problem of $ex(n, F)$ for $\chi(F) \geq 3$ can be reduced to degenerate case.

**Exercise.** Let $V(F) = \{a, b, c, d, e\}$, $E(F) = \{ab, ac, bc, de, cd, ce\}$, prove $ex(n, f) = \lfloor \frac{n^2}{4} \rfloor + 1$

For results and a detailed discussion on decomposition family, we refer interested readers to the survey of Simonovits, "How to solve a Turán type extremal graph problem (linear decomposition)" in 1999.